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LETTER TO THE EDITOR

Exact solutions to quantized field theories

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Abstract. Exact solutions to a class of one dimensional quantum field theories described by the field equations $d^2\phi/dt^2+m^2\phi+\lambda\phi^{2q+1} = 0$, $q \neq 0, -1$, and by boson commutation relations are given. For the special case $q = 1$ explicit solutions are discussed. The solutions describe self-interacting systems for all times and contain arbitrary negative powers of m^2.

In the study of quantum field theories the one dimensional space–time model has recently proved to be quite valuable. In particular, the divergence of the renormalized perturbation series has been established in this model for a $\lambda\phi^4$ self-interaction Hamiltonian (Bender *et al* 1970). In this letter we wish to give some results on solutions of the field equations and boson commutation relations for a class of one dimensional field theories. These theories represent the long wavelength limit of four dimensional space–time theories. The theories given here describe self-interacting systems for all times.

The field equations considered here have the form

$$\frac{d^2\phi_{q\lambda}}{dt^2}+m^2\phi_{q\lambda}+\lambda\phi_{q\lambda}{}^{2q+1} = 0 \tag{1}$$

with $q \neq 0, -1$. Equation (1) may be considered a special case of the Klein–Gordon equation appropriate to spatially homogeneous systems, with the additional self-interaction $\lambda\phi^{2q+1}$. For the self-interaction $\lambda\phi^3$, solutions, and their quantization, without the restriction of spatial homogeneity have been discussed recently (Raczka 1972). The principal characteristic of these solutions is that as t approaches $\pm\infty$ the fields satisfy the Klein–Gordon equation without the self-interaction—they describe free systems. The solutions of equation (1) discussed in this letter have the property that they describe self-interacting systems for all times. Thus, we abandon the unphysical notion of a self-interaction which is somehow 'turned off' in the remote past and in the distant future. In this respect the field theories modelled here lie outside the context of axiomatic field theory (Jost 1965). Physical examples of such intrinsically self-interacting systems include the gravitational field and, perhaps, pi meson fields.

General solutions of equation (1) for certain boundary conditions have been given previously (Reid 1971). A generalization of these solutions to operator valued functions which are particular solutions of equation (1) is

$$\phi_{q\lambda}{}^{(\pm)} = \{1-\lambda(A_{q\lambda}{}^{(\pm)})^{2q}\exp(\mp 2iqmt)/4(q+1)m^2\}^{-1/q}A_{q\lambda}{}^{(\pm)}\exp(\mp imt) \tag{2}$$

as may be verified by direct substitution. It is evident that these solutions oscillate for all times. They also may be characterized as positive frequency or negative frequency fields.

The complete quantum field theory also includes commutation relations on the fields. In the linear theory one method of specifying these commutation relations is to find positive and negative frequency solutions, construct fields ϕ which are linear combinations of $\phi^{(+)}$ and $\phi^{(-)}$ and specify the commutation relation of ϕ and $\mathrm{d}\phi/\mathrm{d}t$. An alternative method is to specify the commutator of $\phi^{(+)}$ and $\phi^{(-)}$. In the nonlinear theory a linear combination of $\phi^{(+)}$ and $\phi^{(-)}$ is no longer a solution of the field equations. Consequently we follow the second method of specifiying the commutator. Thus, we require

$$[\phi_{q\lambda}{}^{(+)}(0), \phi_{q\lambda}{}^{(-)}(0)] = 1 \tag{3}$$

which, as λ approaches zero, implies

$$[A_{q0}{}^{(+)}, A_{q0}{}^{(-)}] = 1. \tag{4}$$

Equation (4) enables us to identify $A_{q0}{}^{(+)}$ and $A_{q0}{}^{(-)}$ with a and $a^\dagger$, the annihilation and creation operators of the linear theory.

For λ different from zero equation (3) leads to an expression for $A_{q\lambda}{}^{(\pm)}$ as a series in λ. Here we will give only the results for $q = 1$. We write

$$A_{1\lambda}{}^{(+)} = \sum_{p=0}^{\infty} \lambda^p a_p \tag{5}$$

with a similar expression for $A_{1\lambda}{}^{(-)}$ in terms of $a^\dagger$. The operators a_p and $a_p{}^\dagger$ are obtained by using the forms for $A_{1\lambda}{}^{(+)}$ and $A_{1\lambda}{}^{(-)}$ in equation (3), expanding in a series in λ and equating coefficients of equal powers of λ. Solutions consistent with equation(3) are

$$a_p = (-\tfrac{1}{8}m^2)^p \alpha_p a^{2p+1} \tag{6}$$

where α_p is a numerical coefficient determined in terms of α_i $(i < p)$ by

$$\begin{aligned} \alpha_p = - \sum_{N=1}^{p} (-1)^N \frac{2N+1}{2p+1} \Bigg(& \sum_{\substack{x_1=0 \\ x_2=1}}^{p-N} \delta_{x_1+x_2,\, p-N} \\ & \times \sum_{\{x_1|Z_i\}} \delta_{A,B} \delta_{F,2N} (2N)! \prod_i \alpha_i{}^{N_i} e_{x_2} \alpha_{x_2} / N_i! \\ & + \sum_{\{p-N|Z_i\}} \delta_{A,p-N} \delta_{F,2N} (2N)! \prod \alpha_i{}^{N_i} / N_i! \Bigg) \end{aligned} \tag{7}$$

with

$$A = \sum_i N_i Z_i \tag{8}$$

$$B = x_1 \tag{9}$$

$$F = \sum_i N_i \tag{10}$$

$$e_{x_2} = 2x_2 + 1 \tag{11}$$

$$\alpha_0 = 1. \tag{12}$$

The expression $\{x|Z_i\}$ denotes a partition of the integer x into a set of integers Z_i

with N_i repetitions. The sum extends over all partitions consistent with the constraints imposed through the Kronecker delta functions. Furthermore, N_0 is determined by partitioning an integer x, counting the elements of the partition and subtracting this number from $2N$. The expression for $a_p{}^\dagger$ is obtained from equation (6) by replacing a by $a^\dagger$.

From the example given here we see that the positive and negative frequency fields contain arbitrary, odd powers of annihilation or creation operators of the linear theory. Another important result is that the fields contain arbitrary negative powers of m^2, the mass associated with the linear theory. Consequently, solutions corresponding to zero mass, if they exist, must be qualitatively different from the solutions found here. Matthews (1970a, 1970b), discussing linear, vector field theories, has also found that zero and nonzero mass must be treated separately.

It is apparent that the method of solution outlined here is applicable for $q \neq 1$. However, the solutions given in this letter are not unique, so we are left with the question of a further physical statement which will limit the possible solutions. This and the generalization of these ideas to four dimensional space–time will be discussed in a subsequent publication.

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